BOUNDARY LAYER ON THIN SHORT WINGS

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The author has analyzed the equations of flow at large Reynolds number over thin short wings. A regular solution has been constructed for vortex-free flow, valid in the linear approximation in the small parameters: angle of attack and wing thickness. It is shown that the three-dimensional boundary layer problem reduces in this case to a set of two-dimensional problems. The necessary equations are given and the analysis is shown. In a comparison with experimental data computed results are shown for laminar and turbulent boundary layers on a triangular wing. The author has investigated some special features of flow on a wing with a bend in the leading edge, accounting for two Reynolds number approximations. The author gives the basic relations and example calculations for a triangular wing.

1. Let a wing of thickness δ_0, length b_0, and chord length $2\ell_0$ be set at a small angle of attack α in a uniform gas stream with density ρ_∞, pressure p_∞, enthalpy h_∞, velocity u_∞, viscosity μ_∞, thermal conductivity k_∞, Mach number M, Reynolds number $\mathrm{Re} = \rho_\infty u_\infty b_0/\mu_\infty$, and Prandtl number Pr. The wing surface is given in a rectangular coordinate system XYZ (Fig. 1) by the equation $Y = \tau F(X, Z)$. The coordinates Y and Z in the transverse section plane are referenced to ℓ_0, and X is referenced to b_0. Assuming that

$$\lambda = l_0/b_0 \ll 1,\ \delta = \delta_0/b_0 \ll 1,\ \tau = \delta/\lambda \ll 1,\ \varepsilon = \alpha/\lambda \ll 1,\ \lambda \mathrm{Re} \gg 1,$$

we seek a solution of the flow equations in the Prandtl approximation (separately in the outer inviscid region and in the boundary layer) using the method of perturbations. With the given boundaries the outer region flow is described by slender body theory [1]. The shortcoming of this solution is the presence of singularities in the vicinity of the wing edge. To obtain a regular solution in this paper we use the method of [2, 3], based on constructing a local solution near the edge, which is matched with the slender body solution. All the relations for the outer region are analytical in form, which eases the analysis considerably.

The solution of the boundary layer equations is also found by the method of matched asymptotic expansions [4, 5]. On the main part of the wing surface the general three-dimensional problem is reduced to a sequence of two-dimensional problems by introducing an additional unknown. In the vicinity of the leading edge the flow is described by the equations for a wing in shear flow [6].

With this technique one can obtain a solution of the flow equations in the following Reynolds number approximation. Some special features of this problem have been studied for a laminar boundary layer on a wing with a discontinuity in its leading edge.

In slender body theory the flow over a wing is described by the potential Φ' which can be represented in the form [1]

$$\Phi' = b_0 u_\infty \{X + \lambda^2[\varphi(Y, Z; X, \tau, \varepsilon) + \varphi_0(X, \tau)] + O(\lambda^4 \ln^2 \lambda)\}, \tag{1.1}$$

$$\varphi_0 = -\frac{1}{2\pi}\frac{d}{dX}\begin{cases}\int\limits_0^X S_\xi(\xi)\ln\frac{2(X-\xi)}{\omega}\,d\xi, & M>1,\\ \int\limits_0^1 S_\xi(\xi)\ln\frac{2|X-\xi|}{\omega}\,d\xi, & M<1,\end{cases}$$

where $\omega = |M^2-1|^{1/2}$; $S(X) = 2\tau\int\limits_{-l}^{l} A(X, Z)\,dZ$ is the transverse section area of the wing; $2A = F_1 + F_2$; F_1 and F_2 are branches of the two-valued function F(X, Z) corresponding to the upper and lower wing surfaces. We shall assume the wing to be symmetrical relative to the plane Z = 0. The equations $Z = \ell(X)$, $Y = \tau C(X, \ell)$ give the position of the wing leading edge, and

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the equations $Y = \tau C(X, Z) = \tau(F_1 - F_2)/2$ give the position of mean surface. The perturbation potential φ is found as a solution of the two-dimensional Laplace equation in the YZ plane with a linearized non-penetration boundary condition [1]. We write the flow parameters in the form

$$\varphi_e = \pm\varepsilon(l^2 - Z^2)^{1/2} \mp \frac{2\tau}{\pi}\int_{-l}^{l} C_X(X,\zeta)\ln\frac{\sqrt{(l-Z)(l+\zeta)}+\sqrt{(l+Z)(l-\zeta)}}{\sqrt{2l}\,|Z-\zeta|}\,d\zeta +$$
$$+\frac{\tau}{\pi}\int_{-l}^{l} A_X(X,\ \zeta)\ln|Z-\zeta|\,d\zeta + O[(\varepsilon+\tau)^2],$$
$$w_e = \pm(l^2 - Z^2)^{-1/2}\left[-\varepsilon Z + \frac{\tau}{\pi}\int_{-l}^{l} C_X\frac{\sqrt{l^2-\zeta^2}}{\zeta - Z}\,d\zeta\right] + \frac{\tau}{\pi}\int_{-l}^{l}\frac{A_X d\zeta}{|\zeta - Z|} +$$
$$+ O[(\varepsilon+\tau)^2],\quad u_e = 1 + \lambda^2(\varphi_{eX} + \varphi_{0X}) = 1 - \lambda^2 p + O(\lambda^4\ln^2\lambda), \tag{1.2}$$
$$v_e = \tau(C_X \pm A_X) + O[(\varepsilon+\tau)^2].$$

Here and below the index e denotes inviscid flow functions on the wing surface, a superscript refers to the upper surface, and a subscript to the lower surface, the projection of the u_e velocity vector on the X axis are referenced to u_∞, the components of the velocity vectors v_e and w_e on the Y and Z axes are referenced to λu_∞, and the pressure perturbations p are referenced to $\lambda\rho_\infty u_\infty^2$. The first terms in the expressions for the potential φ_e and the velocity w_e relate to flow over a flat plate, the second terms account for the curvature of the mean surface, and the third terms relate to the thickness. The solution of Eq. (1.2) has a singularity on the leading edge, and we treat the flow in that vicinity separately.

In the leading edge $Z = \ell$ we fix a system of orthogonal curvilinear coordinates xyz (Fig. 1), where x is measured along the edge, y along the normal to the mean surface, z is measured along the tangent to the mean surface perpendicular to the edge, y and z are referenced to ℓ_0, and x is referenced to b_0. In the framework of slender body theory the angle χ_1 of inclination of the edge relative to the X axis is small: $\chi_1 = \lambda\beta = \lambda l_X + O(\lambda^2) \ll 1$, and the angles of inclination of the mean surface relative to the plane Y = 0 are also small. Therefore the velocity vector components, referenced to u_∞, along the edge are $U_\infty = \cos\chi_1 = 1 + O(\lambda^2)$, and in the direction perpendicular to it the component is $W_\infty = \sin\chi_1 = \lambda\beta + O(\lambda^3)$.

For a blunted edge the characteristic dimension of the special region is referenced to ℓ_0, and the radius of curvature $r_0(x) \ll 1$ of the nose profile of the wing section is a plane orthogonal to the edge. In this region the wing surface, to an accuracy within a higher order of smallness relative to r_0, is approximated by the parabolic surface $y = (2r_0 z)^{1/2} = r_0\sigma$ [σ is the parabolic coordinate, and $z = \ell - Z + O(\lambda^2)$]. The derivatives of the flow functions in the yz plane are on the order $O(r_0^{-1}) \gg 1$, and along the edge they are O(1). Therefore the flow in the region $z = O(r_0)$ is described by a nonlinear two-dimensional equation of the potential in the yz plane. The x coordinate enters only into the boundary conditions as a parameter, and the non-penetration condition is satisfied on the parabolic surface. For $M\lambda \ll 1$ the problem can be linearized and its solution takes the form [2, 7]

$$U_e = U_0(x),\quad W_e = W_0(x)\frac{\sigma - \sigma_0}{H_1},\quad H_1 = (1+\sigma^2)^{1/2}. \tag{1.3}$$

Here W_e is the component, referenced to λu_∞, of the velocity vector on a parabolic wing profile in the plane orthogonal to the edge; U_e is the velocity along the edge, referenced to u_∞. The functions U_0 and W_0 and the position of the line of outflow $\sigma_0(x)$ are found from the solution matching conditions (1.2) and (1.3). Putting $z = \ell$ and letting Z go to 0 in Eq. (1.2) and using the formulas for transforming from Z, Y, Z coordinates, we obtain

$$U_e = u_e\cos\chi_1 + \lambda w_e\sin\chi_1 = 1 + O(\lambda^2),$$
$$\lambda W_e = u_e\sin\chi_1 - \lambda w_e\cos\chi_1 = \lambda\left[\beta + \tau I_1 - \sqrt{\frac{l}{r_0}}\frac{\varepsilon + \tau I_2}{\sigma}\right] + O(\lambda^3), \tag{1.4}$$

where

$$I_1 = \frac{2}{\pi}\int_0^l\frac{A_X\,d\zeta}{l^2-\zeta^2},\quad I_2 = \frac{2}{\pi}\int_0^l\frac{C_X\,d\zeta}{\sqrt{l^2-\zeta^2}}. \tag{1.5}$$

Comparing Eqs. (1.3) and (1.4) as $\sigma \to \infty$, we find

$$U_0 = 1,\ W_0 = \beta + \tau I_1,\ \sigma_0 = -\sqrt{\frac{l}{r_0}}(\varepsilon + \tau I_2). \quad (1.6)$$

For a sharp edge we can also construct a regular solution, if $\varepsilon + \tau I_2 = 0$. Then the local solution in an exponentially small vicinity of the edge is found as a solution of the problem for a wedge in shear flow $y = \pm\psi_0 z$ [($\psi_0(x)$ is the semi-vertex angle of the edge], and has the form

$$U_e = 1,\ W_e = W_0 z^j,\ j = \psi_0/(\pi - \psi_0). \quad (1.7)$$

The effective flow velocity $W_0(x)$ is determined from the condition of matching with the solution (1.2) in the same way as for the profile [3].

2. We seek a solution of the boundary layer equations on the main part of the wing surface, where Eqs. (1.2) are valid, in the form of the asymptotic series

$$\begin{gathered} u = u_0(s,\ n) + \varepsilon u_{10}(s,\ n,\ Z) + \tau u_{11}(s,\ n,\ Z) + \ldots, \\ w = \varepsilon w_{10} + \tau w_{11} + \ldots, \\ h = h_0 + \varepsilon h_{10} + \tau h_{11} + \ldots,\ \rho = \rho_0 + \varepsilon\rho_{10} + \tau\rho_{11} + \ldots, \\ \mu = \mu_0 + \varepsilon\mu_{10} + \tau\mu_{11} + \ldots,\ k = k_0 + \varepsilon k_{10} + \tau k_{11} + \ldots \end{gathered} \quad (2.1)$$

Here $s = X - X_0$; $X_0(Z)$ is the distance from the plane $X = 0$ to the leading edge; n is referenced to $b_0 Re^{-1/2}$ and is normal to the wing surface; u is the component of the velocity vector on the section of the wing profile by the plane Z = const; w is the component of the velocity vector in the direction orthogonal to this profile on the wing surface; v is the velocity normal to the surface; u is referenced to u_∞, w is referenced to λu_∞, v is referenced to $u_\infty Re^{-1/2}$, and ρ, h, μ and k are referenced to their values in the unperturbed flow. We note that in the boundary layer the transverse velocity is of one and the same order, but the perturbations of the remaining flow functions are larger by order $O(\lambda^{-2})$ than the values in the vortex-free flow.

The zero-order approximation to the expansion (2.1) corresponds to flow over a flat plate and does not depend on the transverse coordinate. The functions of the first-order approximation are subject to a linear system of equations similar to that obtained for a wing of finite dimension [4, 5]. Combining the two approximations of Eq. (2.1) the composite solution satisfies the system of equations

$$\begin{gathered} (\rho u)_s + (\rho v)_n - \rho w_s X_{0Z} + \rho q = 0,\ \rho h = 1, \\ \rho[(u - wX_{0Z})u_s + vu_n] = (\mu u_n)_n, \\ \rho[(u - wX_{0Z})h_s + vh_n] - (\gamma - 1)M^2\mu u_n^2 = \left(\frac{k}{\Pr}h_n\right)_n, \\ \rho(uw_s + vw_n) + p_Z - X_{0Z}p_s = (\mu w_n)_n, \\ \rho(uq_s + vq_n) + (p_Z - X_{0Z}p_s)_Z = (\mu q_n)_n, \\ n = 0:\ u = v = w = q = h_n = 0;\ n = \infty:\ u = h = 1,\ w = w_e,\ q = q_e \end{gathered} \quad (2.2)$$

(γ is the adiabatic index, and the subscripts s, n, and Z denote differentiation with respect to the corresponding variables). The first approximation problem even for the composite solution, reduces to two-dimensional if we introduce a new dependent variable $q(s, n, Z) = \partial w/\partial Z$, which is in essence an additional flow integral. The coordinate Z enters Eq. (2.2) as a parameter, and from the geometrical characteristics of the surface the equations contain only the sweepback angle of the leading edge χ such that $X_{0Z} = \lambda\tan\chi = \lambda\cot\chi_1 = 1/\beta$. With an error of $O((\varepsilon + \tau)^2)$ the components of the velocity vector u, v, and w along the curvilinear coordinate axes coincide with the components along the Cartesian coordinate axes X, Y, and Z. In contrast with the wing of finite size [4, 5] Eqs. (2.2) do not have a longitudinal pressure gradient, this being of order $O(\lambda^2)$, and the flow perturbations are due exclusively to three-dimensional effects.

The expansions of Eq. (2.1) and Eq. (2.2) are not applicable in the vicinity of the blunted wing leading edge. Using the ordinary procedure for constructing a local asymptotic solution [3], one can show that in a region of dimension $O(r_0)$ near the edge with an error of $O(r_0)$ the flow is described by the equations

$$\begin{gathered} (\rho W)_\sigma + H_1(\rho V)_N = 0,\ \rho\left(WH_1^{-1}U_\sigma + VU_N\right) = (\mu U_N)_N, \\ \rho\left(WH_1^{-1}W_\sigma + VW_N\right) - H_1^{-1}W_e W_{e\sigma} = (\mu W_N)_N,\ \rho\left(WH_1^{-1}h_\sigma + Vh_N\right) - (\gamma - 1)M^2\mu U_N^2 = \left(\frac{k}{\Pr}h_N\right)_N, \\ N = 0:\ U = V = W = h_N = 0;\ N = \infty:\ U = h = 1,\ W = W_e, \end{gathered} \quad (2.3)$$

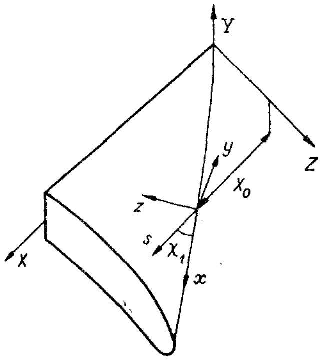

Fig. 1

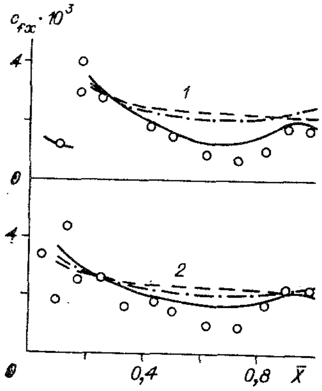

Fig. 2

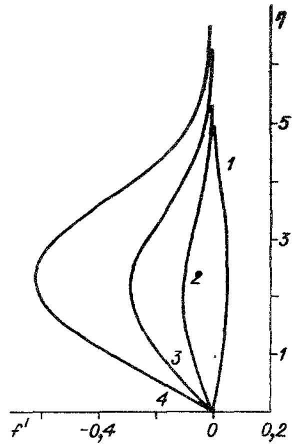

Fig. 3

where $N = n/\sqrt{r_0}$; $V = v\sqrt{r_0}$; U and W are referenced to u_∞ and λu_∞, respectively, along the edge and along the tangent to the parabolic profile of the section of the wing by a plane orthogonal to the edge. We note that the transverse velocity does not enter into the dissipation function of the energy equations (2.2) and (2.3), nor into the relation for the turbulent viscosity μ_t. Thus, in the case of gradient models $\mu_t = d(U_N^2 + \lambda^2 W_N^2)^{1/2} = dU_N + O(\lambda^2)$ [or $\mu_t = d_1 u_n + O(\lambda^2)$ on the main part of the surface]. This property distinguishes Eq. (2.3) from the ordinary equations of theory of a wing shear flow [6].

The solution of the boundary layer problem begins with the outflow line $\sigma = \sigma_0(x)$, where $W_e = 0$, and Eqs. (2.3) reduce to a similarity case. The flow may be both laminar and turbulent, beginning from the outflow line. The location of transition is given or computed using semi-empirical relations.

The boundary layer equations possess the property of being parabolic, and therefore the matching conditions for solutions of Eqs. (2.2) and (2.3) are the initial conditions for Eq. (2.2). It is appropriate to formulate these using the matching principle of Kaplan [3]. Both solutions exist in the intermediate region $r_0 \ll z = r_0\sigma^2/2 = s\beta \ll 1$ and are subject to the same equations for a thin wing in shear flow, which follow from Eq. (2.3) when $\sigma \to \infty$ or from Eq. (2.2) when $s \to 0$. The matching conditions for any point $s_1 = r_0\sigma_1^2/2\beta$ of the intermediate region have the form

$$u(s_1, n, Z) = U(\sigma_1, N, x),\ h(s_1, n, Z) = h(\sigma_1, N, x),$$
$$w(s_1, n, Z) = [W - \beta U]|_{\sigma=\sigma_1},\ q(s_1, n, Z) = -[W_\sigma - \beta U_\sigma]/(r_0\sigma)|_{\sigma=\sigma_1}. \tag{2.4}$$

In the case of a sharp leading edge the formulation of the initial conditions for Eq. (2.2) reduces to solving the well-known similarity problem of laminar flow over a wedge in shear flow, and to a transformation, similar to Eq. (2.4), from one coordinate system to the other.

To illustrate the results obtained Fig. 2 shows the distribution of the longitudinal friction coefficient c_{fx} at the section Z = 0.05 of the top surface of a triangular wing with sharp leading edges, sweepback angle of $\chi = 71.565°$, length $\lambda = 1/3$, thickness $\delta = 0.0843$ ($\tau = 0.253$) at M = 1.5, Re = 10^7, and angles of attack α of 2 and 4° (curves 1 and 2 for $\varepsilon = 0.105$, 0.21). The abscissa is $\bar{X} = s/b(Z)$, where $b = 1 - Z$ is the relative chord at the section Z = const. In the computations we used the Cebesi–Smith turbulent viscosity model and the method of [8]. Near the wing edge there were sections of laminar flow (Fig. 2, $\alpha = 2°$, $\bar{X} \leqslant 0.15$), and the location of transition was taken from experiment [9]. The components of the velocity vector at the outer edge of the boundary layer were computed using the linear equations of slender wing theory with the known experimental pressure distribution [9]. Splines were used to match the experimental data and the computed pressure derivatives. The method of strips (the dot–dash lines), in which the equations of a planar boundary layer are solved along each section, did not lead to an appreciable difference in the results from the zero approximation case (broken lines), while the composite solution (solid curves) agrees well qualitatively with the experimental data of [9] (circles), although the perturbation parameters are relatively large in this case. This confirms the above conclusion as to the influence of three-dimensional effects.

3. As another example we consider laminar flow over a thermally insulated thin wing with a bent leading edge, accounting for the second approximation in Reynolds number for $Pr = 1$, $\alpha = 0$. The wing is a flat plate composed of two triangles such that the wing vertex angle is $2\chi_0 = 2\lambda\beta_0$, and at the bend point $(X_1, 0, \pm Z_1)$ the angle between the X axis and the edge varies up to $\chi_1 = \lambda\beta_1$; we note that the wing vertex is also a leading edge bend point. The coordinate s is given by the relations

$$s = \begin{cases} X - |Z|/\beta_0, & |Z| < Z_1, \\ X - X_1 - (|Z| - Z_1)/\beta_1, & |Z| > Z_1. \end{cases}$$

In this problem the expansion parameter is $\tau_1 = Re^{-1/2}/\lambda$, and the zero approximation is described by the similarity solution

$$u_0 = f_0'(\eta),\ h = t_0'(\eta),\ n = \sqrt{s}\int_0^\eta h_0\,d\eta = \sqrt{s}\,t_0(\eta) \tag{3.1}$$

(f_0 and t_0 are the stream function and the initial enthalpy, and the primes denote differentiation with respect to η). The planes $Z = 0, \pm Z_1$ for this solution are surfaces of weak discontinuities, since

$$u_{0Z} = -\frac{ns^{-3/2}}{2h_0}f_0''\begin{cases} 1/\beta_0, & Z = +0 \\ -1/\beta_0, & Z = -0 \end{cases} = -\frac{ns^{-3/2}}{2h_0}f_0''\begin{cases} 1/\beta_0, & Z = Z_1 - 0, \\ 1/\beta_1, & Z = Z_1 + 0. \end{cases}$$

The same discontinuities occur in the enthalpy and the displacement thickness $\delta^* = as^{1/2}$ ($a = Re^{-1/2}(1.721 + 1.192(\gamma - 1)M^2)$ for $Pr = 1$ [10]).

The perturbations inserted into the potential flow by the boundary layer of such a wing are determined by Eq. (1.2) in which one must put $\varepsilon = C = 0$, $A = s^{1/2}$, $\tau = a/\lambda$. For a triangular wing ($\beta_0 = \beta_1 = 1$) we obtain

$$\Phi_e = \varphi_e + \varphi_0 = \frac{2\tau}{\pi}\sqrt{X}\left[Q(\zeta) + \ln\frac{\omega}{8} + \begin{cases} 0, & M > 1 \\ \frac{1}{2}\ln\frac{1+\sqrt{X}}{1-\sqrt{X}} - \frac{1}{\sqrt{X}}, & M < 1 \end{cases}\right],$$

$$w_e = \Phi_{eZ} = \frac{2\tau}{\pi}\left\{\ln|\zeta|\frac{\sqrt{1+\zeta}-\sqrt{1-\zeta}}{2\sqrt{1-\zeta^2}} + \frac{\ln(1+\sqrt{1+\zeta})}{\sqrt{1+\zeta}} + \frac{\ln(1+\sqrt{1-\zeta})}{\sqrt{1-\zeta}}\right\}, \tag{3.2}$$

where $\zeta = Z/X$; $Q = R(\zeta) + R(-\zeta)$; $R = (1/2)(1 - \sqrt{1+\zeta})\ln|\zeta| + \sqrt{1+\zeta}\ln(1 + \sqrt{1+\zeta})$. For the perturbations of velocities w_e and $u_{1e} = \Phi_{eX}$ there are no singularities on the leading edges, apart from the wing vertex $X = 0$. For $M < 1$ the function u_{1e} has a logarithmic singularity at the trailing edge $X = 1$. In the symmetry plane $Z = 0$ there is a logarithmic singularity in the derivative w_{eZ}, but all the remaining functions are regular. For $\zeta \to 0$ we obtain

$$\frac{\pi}{2\tau}\sqrt{X}\,w_e \approx -\zeta\ln|\zeta|,\ \frac{\pi}{\tau}X^{3/2}w_{eX} \approx -\frac{3}{2}\zeta\ln|\zeta|,\ \frac{\pi}{2\tau}X^{3/2}w_{eZ} \approx \ln|\zeta| - 2\ln 2 + 2.$$

In the case of a wing with fractured edges for $M > 1$ and $X < 1$

$$\Phi_e = \frac{2\tau\beta_0}{\pi}\sqrt{X}\left[Q\left(\frac{\zeta}{\beta_0}\right) + \ln\frac{\beta_0\omega}{8}\right],\ w_e = \frac{2\tau}{\pi}X^{-1/2}Q'\left(\frac{\zeta}{\beta_0}\right).$$

For $X > X_1$ the perturbation potential Φ_e and the transverse velocity are determined by the relations

$$\Phi_e = \frac{2\tau}{\pi}\Big\{\beta_0\sqrt{X}\left[Q\left(\frac{\zeta}{\beta_0}\right) + \ln\frac{\beta_0\omega}{8} + 2\ln(\sqrt{X} + \sqrt{X_1})\right] -$$
$$-\beta_0(\sqrt{X} - \sqrt{X_1})\ln(X - X_1) + \sqrt{X - X_1}\left[\beta_1\left[R\left(\frac{\zeta_1}{\beta_1}\right) + R\left(-\frac{\zeta_2}{\beta_1}\right)\right] -\right.$$
$$\left.-\beta_0\left[R\left(\frac{\zeta_1}{\beta_0}\right) + R\left(-\frac{\zeta_2}{\beta_0}\right)\right] + (\beta_1 - \beta_0)\ln\frac{\beta_1\omega}{8} + \beta_0\ln\frac{\beta_1}{\beta_0}\right] - \beta_0 X_1^{1/2}\ln 4X_1\Big\},$$

$$w_e = \frac{2\tau}{\pi}\left\{X^{-1/2}Q'\left(\frac{\zeta}{\beta_0}\right) + (X - X_1)^{-1/2}\left[R'\left(\frac{\zeta_1}{\beta_1}\right) - R'\left(-\frac{\zeta_2}{\beta_1}\right) - R'\left(\frac{\zeta_1}{\beta_0}\right) + R'\left(-\frac{\zeta_2}{\beta_0}\right)\right]\right\},$$

where the primes denote differentiation with respect to the argument, and $\zeta_1 = (Z + Z_1)/(X - X_1)$, $\zeta_2 = (Z - Z_1)/(X - X_1)$. It can be seen that the bend point is the same type of singular point as the wing vertex. In the symmetry plane $Z = 0$ and in the planes $Z = \pm Z_1$ the derivative w_{eZ} has a logarithmic singularity.

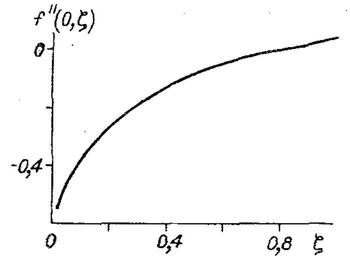

Fig. 4

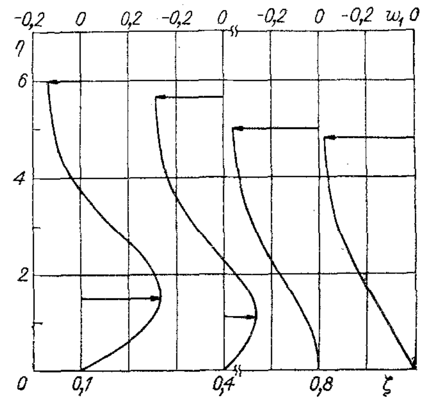

Fig. 5

Using Eq. (3.2) we represent the solution of the boundary layer equation in the next Reynolds number approximation for the triangular wing in the form

$$w = \tau w_{11} = w_e g'(\eta, \xi),\ u_{11} = \frac{P(\zeta)}{\pi\sqrt{X}} f'(\eta, \xi),\ h_{11} = \frac{P(\zeta)}{\pi\sqrt{X}} t'(\eta, \xi),$$

$$\xi = 1 - \zeta,\ P(\zeta) = \left(1 - (1+\zeta)^{-1/2} - (1-\zeta)^{-1/2}\right)\ln|\zeta| + \frac{\ln(1+\sqrt{1+\zeta})}{\sqrt{1+\zeta}} + \frac{\ln(1+\sqrt{1-\zeta})}{\sqrt{1-\zeta}}.$$

The equations for the functions f, g, and t are obtained from Eq. (2.2) after they are linearized relative to the zero approximation of Eq. (3.1) and for $\rho\mu$ = Pr = 1 have the form

$$\begin{gathered} f''' = -0{,}5 f_0 f'' + \xi(1-\xi) f_0' f_\xi' - \xi B f_0' f' - D f_0'', \\ t''' = -0{,}5 f_0 t'' + \xi(1-\xi) f_0' t_\xi' - \xi B f_0' t' - D t_0'' - 2(\gamma - 1) M^2 f_0'' f'', \\ g''' = -0{,}5 f_0 g'' + \xi(1-\xi) f_0' g_\xi' + 0{,}5\xi\left(1 + 2(1-\xi) Q_{\zeta\zeta}/Q_\zeta\right)\left(t_0' - f_0' g'\right), \\ \eta = 0\colon f = t = g = f' = t'' = g' = 0;\ \eta = \infty\colon f' = t' = 0,\ g' = 1, \end{gathered} \tag{3.3}$$

where the coefficients B and D are determined by the expressions

$$B = 0{,}5 + (1-\xi)P_\zeta/P,\ D = \xi(1-\xi)f_\xi - (1-\xi)(\xi P_\zeta/P - 0{,}5)f + [0{,}5(2\xi Q_{\zeta\zeta} - Q_\zeta)g - \xi Q_\zeta g_\xi]/P.$$

In the problem considered the variables are separated and one does not require a supplemementary equation for the function q, since

$$q = w_Z = (w_e g')_Z = \left(w_{e\zeta} g' - w_e g_\xi'\right)/X.$$

Equations (3.3) were solved using the numerical technique of [11] for M = 2. Graphs of the function $f'(\eta, \xi)$ are shown in Fig. 3, the numbers 1-4 pertaining to ξ = 0,1, 0.5, 0.7, and 0.9. Figure 4 shows a graph of the function $f''(0, \zeta)$. The perturbations of the longitudinal velocity u_{11} and enthalpy h_{11}, like the functions f' and t', have logarithmic singularities in the symmetry plane Z = 0 (ξ = 1). The nature of the flow in the transverse plane is illustrated in Fig. 5, which has graphs of the function $w_1 = (\pi\sqrt{X}/2\tau)w = Q_\zeta g'(\eta, \xi)$. It can be seen that for $0 \leqslant \xi \leqslant 0.2$ the perturbations are transmitted from the edge to the plane of symmetry. For $\xi > 0.2$ in the outer part of the boundary plane, but inside the boundary layer there is a region where the transverse velocity is directed in the opposite sense.

4. The external flow model considered in Sec. 1 is simple and allows a visible analytical form of representing the solution, but its region of application is restricted to subsonic and low supersonic flow velocities. The solution for the boundary layer of Eqs. (2.1)-(2.3) is valid near sonic speed and also in the hypersonic range of application of the theory of small perturbations for the outer flow.

Thus, for $M \gg 1$, $M\delta \ll 1$, $M\lambda \geqslant O(1)$, $\lambda \leqslant \chi_1 \leqslant 1$, if the shock waves are attached to the edges, the following estimates hold [12]:

$$\rho = O(1),\ h = O(1),\ p \sim w \sim q \sim \delta/(M\lambda^2) = \tau/(M\lambda).$$

For $\delta/(M\lambda^2) \ll 1$ Eqs. (2.1) and (2.2) are applicable to describe the hypersonic boundary layer over the entire wing surface. When the shock waves are detached the estimates are different [13]:

$$\rho = O(M^{-2}),\ h = O(M^2),\ p \sim w \sim q \sim \delta M/\lambda^2 = \tau M/\lambda,$$

so that Eqs. (2.1) and (2.2) are valid on the main part of the surface, if $\delta M/\lambda^2 \ll 1$. The flow near the edge in this case is described by the Euler equations and the boundary layer equations in the wing in shear flow approximation. In the intermediate region $\delta M/\lambda^2 = O(1)$ the boundary layer on the wing is described by the full equations of the three-dimensional boundary layer [13], and for $\delta M/\lambda^2 \gg 1$ another limiting solution is valid [14].

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